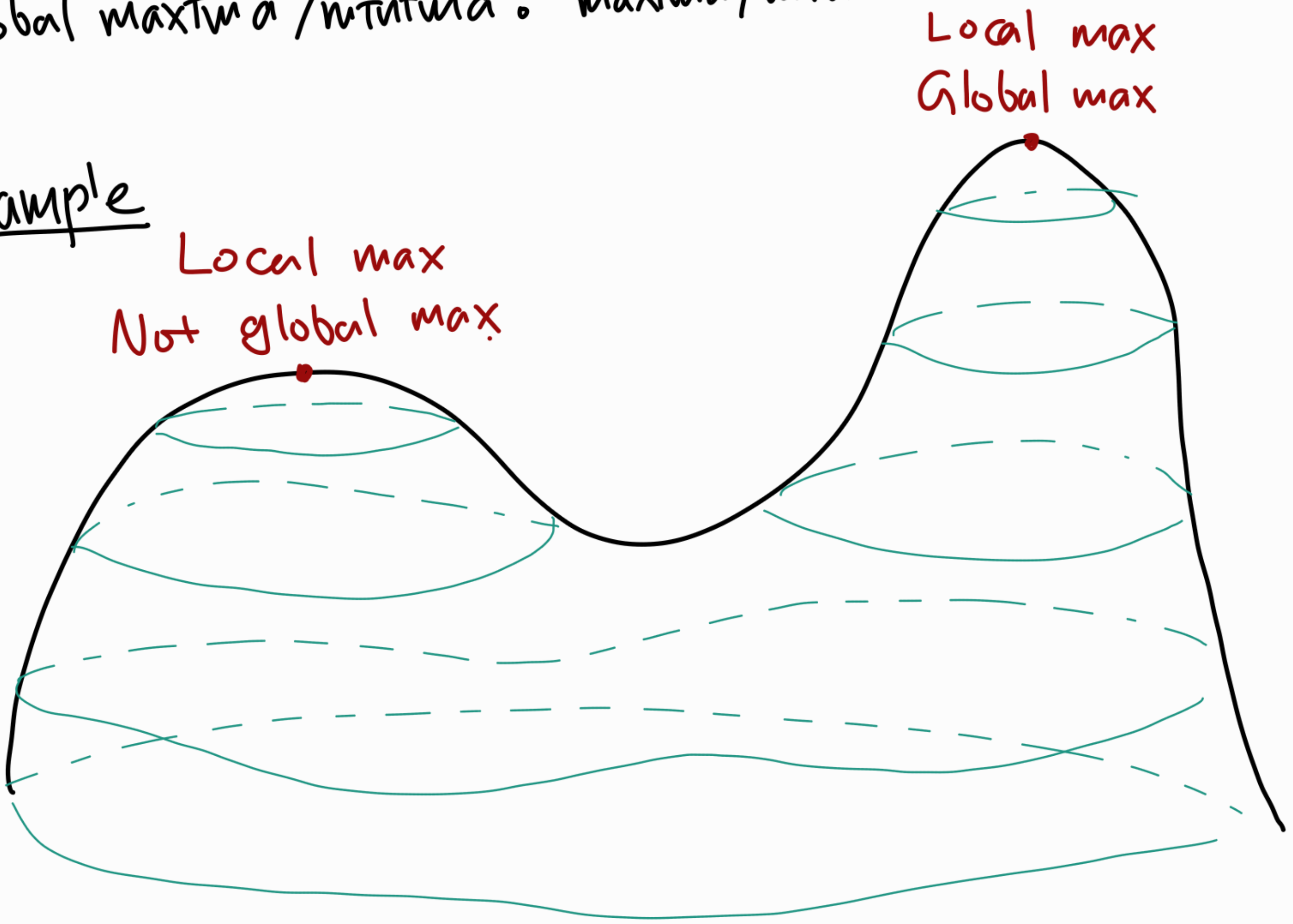
Global maxima/minima (a.k.a. absolute maxima/minima)

Local maxima/minima: maxima/minima near the point.

Global maxima/minima: maxima/minima of all.

Example



The basic tool for finding the global max/min:

Theorem On a compact domain, a global max/min is either a critical point or on the boundary of the domain.

⚠ Global maximum/minimum refer to the locations

Global maximum/minimum values refer to the values

Example (Single-variable case)

Find the global maximum and minimum values of $f(x) = x^3 - 3x^2 + 1$ on the domain $-\frac{1}{2} \le x \le 4$.

Solution We study 2 types of points.

Ⓐ Critical points

x is a critical point if $f'(x) = 0$.

$f'(x) = 3x^2 - 6x$, so

$f'(x) = 0$ means

$$3x^2 - 6x = 0.$$

$$\Updownarrow$$

$$3x(x-2) = 0$$

$$\Updownarrow$$

$x = 0$ or $x = 2$.

Critical points are $x=0, x=2$.

Ⓑ Boundary points.

Domain is $-\frac{1}{2} \le x \le 4$.

Boundary points are

$x = -\frac{1}{2}$, $x = 4$.

Type	x	$f(x)$.
Critical	0	$0^3-3\cdot 0^2+1=\boxed{1}$
	2	$2^3-3\cdot 2^2+1=\boxed{-3}$
Boundary	$-\frac{1}{2}$	$\left(-\frac{1}{2}\right)^3-3\cdot\left(-\frac{1}{2}\right)^2+1=\boxed{\frac{1}{8}}$
	4	$4^3-3\cdot 4^2+1=\boxed{17}$

Global max value = Largest of above = $\boxed{17}$

Global min value = Smallest of above = $\boxed{-3}$

Let's think about the two-variable case.

<u>Example</u> Want: Global max/min values of

$f(x,y) = x+y$ on the domain $x^2+y^2 \le 1$.

We again look for 2 types of points.

Ⓐ Critical points. Ⓑ Boundary points.

Ⓐ Critical points.

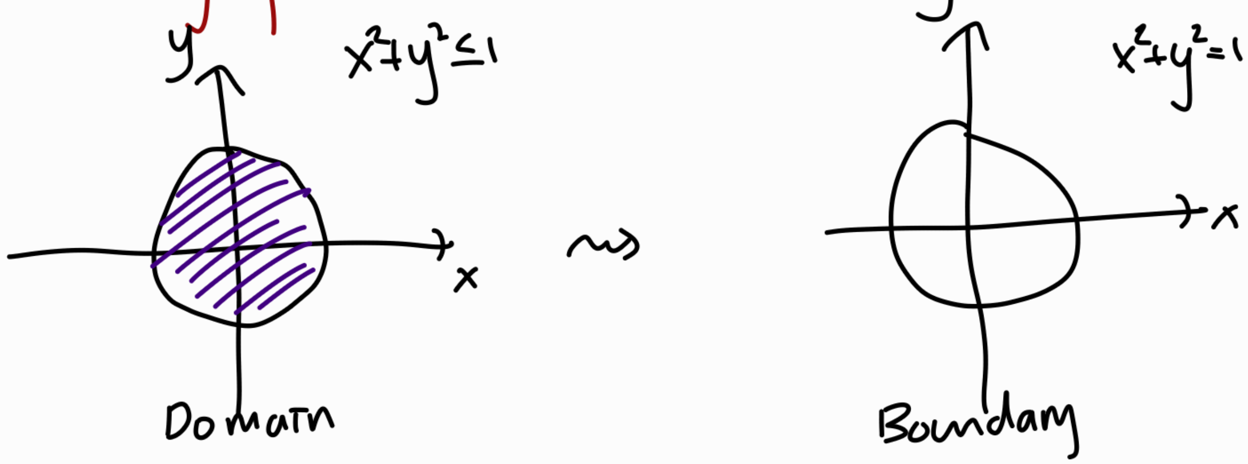
(x,y) is a critical point of $f(x,y)$ if

$\nabla f(x,y) = \langle 0,0 \rangle$.

Note $\nabla f(x,y) = \langle 1,1 \rangle$, so it's never $\langle 0,0 \rangle$.

$\Rightarrow$ No critical points

Ⓑ Boundary points.



The boundary is the circle $x^2+y^2=1$.

<u>Problem</u> There are still ∞ many points to consider for boundary points.

Single-variable: Boundary points are a bunch of points.

Two-variable: Boundary points form a curve, still ∞ many possibilities.

⇨ Need to know:

What is the max/min value of $f(x,y)=x+y$ on the domain $x^2+y^2=1$?

This problem is answered precisely by the method of <u>Lagrange multipliers</u>.

Lagrange multipliers :

Global max/min problem when the domain has equality (=)

⇨ Tells you how to find Lagrange critical points

No equalities ⇨ Critical + Boundary points
Equalities ⇨ Lagrange critical + Boundary points.

Example

Global max/min of $f(x,y) = x+y$ on the domain $x^2+y^2 \leq 1$: NOT Lagrange multipliers

Global max/min of $f(x,y) = x+y$ on the domain $x^2+y^2 = 1$: Lagrange multipliers

There is a general version of Lagrange multipliers with complicated domains, but today we will learn a baby version, 2 variables & one equality.

Lagrange multipliers, 2 variables, 1 equality version

Setup: Find global max/min of $f(x,y)$,
on the domain $g(x,y)=k$.

Method of Lagrange multipliers.

A global max/min of $f(x,y)$ is
either a Lagrange critical point or a boundary point.

Lagrange critical point (2 variable, 1 equality)

(x,y) is a Lagrange critical point if
the two vectors $\nabla f(x,y)$ and $\nabla g(x,y)$ are
parallel to each other.

This happens if
EITHER 1 $\nabla g(x,y) = \langle 0,0 \rangle$
OR 2 There is a scalar λ such that
$\nabla f(x,y) = \lambda \nabla g(x,y)$.

Boundary point (2 variable, 1 equality)

In the case of 2 variable, 1 equality version,
there are no boundary points.

Example Find the global max/min values of

$f(x,y) = x+y$ on the domain $x^2+y^2=1$.

Solution. The domain $\{x^2+y^2=1\}$ is expressed in terms of 1 equality.

So this is 2 variable, 1 equality case

$\Rightarrow$ Lagrange multipliers.

$f(x,y)=x+y$ on domain $g(x,y)=1$

(we set $g(x,y)=x^2+y^2$).

Lagrange critical points if

EITHER ① $\nabla g(x,y) = \langle 0,0\rangle$

OR ② There is a scalar λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y).$$

$$\boxed{\text{Case ① } \nabla g(x,y) = \langle 0,0\rangle}$$

$g(x,y) = x^2+y^2 \Rightarrow \nabla g(x,y) = \langle 2x, 2y\rangle$

So $\nabla g(x,y) = \langle 0,0\rangle$ means $\langle 2x,2y\rangle = \langle 0,0\rangle$

$\Rightarrow (x,y)=(0,0)$.

However, $(x,y)=(0,0)$ does not satisfy $x^2+y^2=1$

$\Rightarrow$ no Case 1 Lagrange critical point on the domain

Case 2 There is a scalar λ such that

$$\nabla f(x,y)=\lambda \nabla g(x,y)$$

$f(x,y)=x+y \Rightarrow \nabla f(x,y)=\langle 1,1\rangle$

$g(x,y)=x^2+y^2 \Rightarrow \nabla g(x,y)=\langle 2x,2y\rangle$

so $\nabla f(x,y)=\lambda \nabla g(x,y)$ is

$$\langle 1,1\rangle=\lambda\langle 2x,2y\rangle=\langle 2\lambda x, 2\lambda y\rangle.$$

$\Rightarrow$ System of equations

$1=2\lambda x \Rightarrow x=\frac{1}{2\lambda}$ (λ is never 0)

$1=2\lambda y \Rightarrow y=\frac{1}{2\lambda}$ (λ is never 0)

CAUTION!!

When you want to divide by something, make sure you always check if it's possible for the denominator to be zero!!!

$\Rightarrow$ $x=\frac{1}{2\lambda}=y$. Since $x^2+y^2=1$, this means $2x^2=1$.

$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \frac{1}{\sqrt{2}}$ OR $x = -\frac{1}{\sqrt{2}}$.

We have $y = x$, so the Case 2 Lagrange critical points are $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ OR $(x,y) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

In total: Lagrange critical points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Type	(x,y)	$f(x,y)$
Lagrange critical	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \boxed{\sqrt{2}}$
	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = \boxed{-\sqrt{2}}$
Boundary	N/A	

Global max value of $f(x,y)$ on the domain $g(x,y) = 1$: $\boxed{\sqrt{2}}$

Global min value of $f(x,y)$ on the domain $g(x,y) = 1$: $\boxed{-\sqrt{2}}$

Back to the original question:

Global max/min values of $f(x,y) = x+y$ on the domain $x^2+y^2 \le 1$.

Type		(x,y)	$f(x,y)$
Critical		N/A	
Boundary: Global max/min of $f(x,y)$ on domain $x^2+y^2=1$	Lagrange critical	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \boxed{\sqrt{2}}$
		$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right) = \boxed{-\sqrt{2}}$
	Boundary	N/A	

Global max value of $f(x,y)$ on domain $x^2+y^2 \le 1$: $\boxed{\sqrt{2}}$

Global min value of $f(x,y)$ on domain $x^2+y^2 \le 1$: $\boxed{-\sqrt{2}}$

Example Find the global max/min values of

$$f(x,y) = x^2 + 2y^2 \quad \text{on the domain } g(x,y) = x^2 + y^2 \le 1.$$

Solution. As before, we have the following steps to take.

- Max / Min
  - Ⓐ Critical ($x^2+y^2 \le 1$)
  - Boundary ($x^2+y^2 = 1$)
    - Ⓑ Lagrange critical ($x^2+y^2=1$)
    - Boundary (We know no points are here!)

Case Ⓐ Critical points (Domain: $x^2+y^2 \le 1$)

This happens when $\nabla f(x,y) = \langle 0,0 \rangle$.

$$f(x,y) = x^2 + 2y^2 \Rightarrow \nabla f(x,y) = \langle 2x, 4y \rangle$$

so, $\nabla f(x,y) = \langle 0,0 \rangle$ means $\langle 2x, 4y \rangle = \langle 0,0 \rangle$,

so $x=0$, $y=0$.

$(x,y) = (0,0)$ satisfies $x^2+y^2 \le 1$

$\Rightarrow \boxed{(0,0)}$ is a critical point.

Case B Lagrange critical points (Domain: $x^2+y^2=1$)

This happens when

EITHER 1 $\nabla g(x,y) = \langle 0,0\rangle$

OR 2 There is a scalar λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y).$$

Case B-1 $\nabla g(x,y) = \langle 0,0\rangle$ (Domain: $x^2+y^2=1$)

$$g(x,y) = x^2+y^2 \Rightarrow \nabla g(x,y) = \langle 2x, 2y\rangle$$

So, $\nabla g(x,y) = \langle 0,0\rangle$ means $\langle 2x,2y\rangle = \langle 0,0\rangle$

$$\Rightarrow (x,y) = (0,0)$$

However, $(x,y)=(0,0)$ does not satisfy $x^2+y^2=1$

$\Rightarrow$ No Lagrange critical point on the domain in Case B-1

Case B-2 There is a scalar λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$ (Domain: $x^2+y^2=1$)

$$f(x,y) = x^2+2y^2 \Rightarrow \nabla f(x,y) = \langle 2x, 4y\rangle$$

$$g(x,y) = x^2+y^2 \Rightarrow \nabla g(x,y) = \langle 2x, 2y\rangle$$

So, $\nabla f(x,y) = \lambda \nabla g(x,y)$ is $\langle 2x, 4y\rangle = \lambda\langle 2x,2y\rangle = \langle 2\lambda x, 2\lambda y\rangle$.

⇨ system of equations

$$2x = 2\lambda x \quad \dots \text{ Equation (a)}$$

$$4y = 2\lambda y. \quad \dots \text{ Equation (b)}$$

What are the solutions?

Equation (a) $2x = 2\lambda x$ has 2 possible consequences.

If $2x \neq 0$, we can divide Equation (a) by $2x$, and get $\boxed{\lambda = 1.}$

$\boxed{\lambda = 1}$ means Equation (b) becomes $4y = 2y$, so $\boxed{y = 0.}$

Our domain is $x^2 + y^2 = 1$, so $y = 0$ means $x^2 = 1$, so either $\boxed{x = 1}$ or $\boxed{x = -1.}$

Lagrange critical points

$(x, y) = \boxed{(1, 0)}$ or $\boxed{(-1, 0).}$

If $2x = 0$, $\boxed{x = 0.}$

Our domain is $x^2 + y^2 = 1$, so $\boxed{x = 0}$ means $y^2 = 1$, so either $\boxed{y = 1}$ or $\boxed{y = -1.}$

Lagrange critical points

$(x, y) = \boxed{(0, 1)}$ or $\boxed{(0, -1)}$

Finally:

Types		(x,y)	$f(x,y)$
Critical (Domain: $x^2+y^2\le 1$)		$(0,0)$	$0^2+2\cdot 0^2=\boxed{0}$
Boundary (Domain: $x^2+y^2=1$)	Lagrange critical	$(1,0)$	$1^2+2\cdot 0^2=\boxed{1}$
		$(-1,0)$	$(-1)^2+2\cdot 0^2=\boxed{1}$
		$(0,1)$	$0^2+2\cdot 1^2=\boxed{2}$
		$(0,-1)$	$0^2+2\cdot(-1)^2=\boxed{2}$
	Boundary	N/A	

Global max value of $f(x,y)$ on domain $x^2+y^2\le 1$: $\boxed{2}$

Global min value of $f(x,y)$ on domain $x^2+y^2\le 1$: $\boxed{0}$